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EXTENDED ABSTRACT*

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Abstract. We report work in progress concerning the theoretical basis and the implementation in the Theorema system of a methodology for the generation of verification conditions for recursive procedures, with the aim of practical verification of recursive programs. Proving total correctness is achieved by proving separately partial correctness and then termination. In contrast to other approaches, which use a special theory describing the behavior of programs, we use such a theory only "in the background", for developing a general rule for generating verification conditions, while the conditions themselves are presented (and provable) using the theories relevant to the program text only. This is very important for automatic proving, since it reduces significantly the effort of the provers. We present practical experiments in which several programs are completely verified using our verification condition generator and the provers of the Theorema system.

Introduction. While proving [partial] correctness of non-recursive procedural programs is quite well understood, for instance by using Hoare Logic [3], [5], there are relatively few approaches to recursive procedures (see e.g. [7] Chap. 2).

We discuss here a practical approach to automatic generation of verification conditions for functional recursive programs, based on a Scott inductive theory [12,10,6,8] and its implementation. The implementation is part of the *Theorema* system, and complements the research performed in the *Theorema* group on verification and synthesis of functional algorithms based on logic principles [1,2,4].

The *Theorema* system (`www.theorema.org`, [11]) aims at realization of a computer aided assistant for the working mathematicians and engineers, which integrates automatic reasoning, algebraic computing, and equational solving. The system provides an uniform environment in natural logico-mathematical language for defining, testing, and proving properties of algorithms, and in general for creating and investigating mathematical models.

We consider the correctness problem expressed as follows: *given* the program (by its source text) which computes the function F and given its specification by a precondition on the input $I_F[x]$ and a postcondition on the input and the output $O_F[x, y]$, *generate* the verification conditions which are [minimally] sufficient for the program to satisfy the specification.

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For simplifying this presentation, we consider functions from $\mathbb{D} \to \mathbb{D}$. The functional program of F can be interpreted as a set of predicate logic formulae, and the correctness of the program can be reduced to proving:

$$(\forall x \in \mathbb{D})\ (I_F[x] \Longrightarrow O_F[x, F[x]]),$$

which we will call the *correctness formula* of F. The proof uses the formulae corresponding to the definition of the function and the theories describing the properties of the predicates and the functions occurring in the program text. This approach was previously used by other authors and is also experimented in the Theorema system [1]. However, the proof of such one-single theorem may be difficult, because the prover has to find the appropriate induction principle and has to find out how to use the properties of the auxiliary functions present in the program.

The method presented in this paper generates several simpler verification conditions, which are easier to prove. In particular, only the termination condition needs an inductive proof, and this termination condition is "reusable", because it basically expresses an induction principle which may be useful for several programs. This is important for automatic verification, in particular if this is embedded in a practical verification system, because it leads to early detection of bugs, when proofs of simpler conditions fail.

Moreover, the verification conditions are provable in the frame of predicate logic, without using any theoretical model for program semantics or program execution, but only using the theories relevant to the predicates and functions present in the program text. This is again important for the automatic verification, because any additional theory present in the system will significantly increase the proving effort.

We start by developing a set of rules for generating verification conditions, for programs having a particular structure. The rules are developed using Scott induction, however the verification conditions themselves do not refer to this theory, they only state facts about the predicates and functions present in the program text. In particular, the termination condition consists in the correctness of a certain simplified version of the original program. By inspecting the shape of these rules for several program structures, it is possible to derive a more general rule for the derivation of verification conditions, such that the correctness formula (see above) **is a logical consequence of these verification conditions** in the frame of predicate logic, without using the Scott induction model.

We approach the correctness problem by splitting it into two parts: *partial correctness* (prove that the program satisfies the specification provided it terminates), and *termination* (prove that the program always terminates). Proving *partial correctness* may be achieved by Scott's induction [12,10,6,8] – a detailed description of the method for a certain class of functional programs is presented in [9].

Example: Simple Recursive Programs. We give here the formalization corresponding to a simple recursive structure. Given a domain $\mathbb{D}$, let be the program for a function F:

$$(\forall x \in \mathbb{D})\ F[x] = \ \textbf{If}\ Q[x]\ \textbf{then}\ S[x]\ \textbf{else}\ C[x, F[R[x]]],$$

where Q is a predicate on $\mathbb{D}$ and S, C, and R are auxiliary functions whose total correctness is assumed (S is a "simple" function whose definition does not use F, C is a "combinator" function, and R is a "reduction" function). Let $I_F[x], O_F[x, y]$ be the precondition, respectively the postcondition of F, and similarly let $I_S[x], O_S[x, y], I_C[x, y], O_C[x, y, z], I_R[x], O_R[x, y]$ be the preconditions and the postconditions of the auxiliary functions (we assume the the correctness formula holds for each of them).

Note that the program above can be seen as an abbreviated notation for the logical formulae:

$$(\forall x \in \mathbb{D})\ (Q[x] \implies F[x] = S[x])$$
$$(\forall x \in \mathbb{D})\ (\neg Q[x] \implies F[x] = C[x, F[R[x]]]).$$

Using the Scott induction model, one obtains the following verification conditions for the *partial correctness* of the function F :

$$(\forall x \in \mathbb{D})\ (I_F[x] \wedge Q[x] \implies I_S[x])$$
$$(\forall x \in \mathbb{D})\ (I_F[x] \wedge Q[x] \implies O_F[x, S[x]])$$
$$(\forall x \in \mathbb{D})\ (I_F[x] \wedge \neg Q[x] \implies I_R[x])$$
$$(\forall x \in \mathbb{D})\ (I_F[x] \wedge \neg Q[x] \implies I_F[R[x]])$$
$$(\forall x, y \in \mathbb{D})\ (I_F[x] \wedge \neg Q[x] \wedge O_F[R[x], y] \implies I_C[x, y])$$
$$(\forall x, y \in \mathbb{D})\ (I_F[x] \wedge \neg Q[x] \wedge O_F[R[x], y] \implies O_F[x, C[x, y]])$$

Termination of the program needs an additional condition. This can be expressed using a "simplified" version of the initial function:

$$(\forall x \in \mathbb{D})\ F'[x] = \textbf{ If } Q[x] \textbf{ then } 0 \textbf{ else } F'[R[x]],$$

which only depends on Q and R. We show that, in order to prove termination of the function F, it is sufficient to prove that $(\forall x \in \mathbb{D})\ F'[x] = 0$.

An alternative to this is to prove:

$$((\forall x \in \mathbb{D})\ ((Q[x] \implies P[x]) \wedge (\neg Q[x] \implies (P[R[x]] \implies P[x])))) \implies (\forall x \in \mathbb{D}) P[x],$$

where P is a new predicate symbol, which is a proof in second order logic of the formula above for any P.

Both conditions define in fact an induction principle, which, in conjunction with the partial correctness verification conditions, have as logical consequence the correctness formula. Namely, by taking $P[x] \iff (I_F[x] \implies O_F[x, F[x]]$, one can prove the correctness formula for F, using the partial correctness verification conditions and the correctness formulae for the auxiliary functions.

Note that both conditions only depend on Q and R, thus they abstract some part of the function definition. In practical programming, these Q and R correspond to typical

algorithm schemes (take e. g. $Q[x] \Longleftrightarrow (x = 0)$ and $R[x] = (x - 1)$), thus the termination condition (whose proof usually involves induction) is usable for several programs.

Generalization. By applying the same method to other recursion schemes, one notes that the verification conditions for the partial correctness can be generated directly (without using Scott induction) by applying few basic principles:

- check the input conditions when calling subroutines;
- accumulate all reasonable assumptions (coming from the input condition of the main function, from **if**–**then**–**else** statements and from the correctness formulae of the subroutines),
- try to finally obtain the correctness property for the output of F.

Furthermore, the termination condition can be generated as the appropriate induction principle which makes the correctness formula a logical consequence of all the verification conditions.

Implementation and experiments. The methods described above are implemented in the *Theorema* system and we are studying various test cases in order to improve the power of our verification condition generator. Furthermore, the concrete proof problems are used as test cases for our provers and for experimenting with the organization of the mathematical knowledge. Currently we are able to generate the verification conditions and to prove them automatically in the Theorema system for various concrete programs.

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