On Hadwiger's Number of a graph with partial information

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Abstract

We investigate the possibility of proving upper bounds on Hadwiger's number of a graph with partial information, mirroring several known upper bounds for the chromatic number. For each such bound we determine whether the corresponding bound for Hadwiger's number holds. Our results suggest that the "locality" of an inequality accounts for the existence of such an extension.

Key words: Hadwiger's number, upper bounds.

1 Introduction

The celebrated Hadwiger's conjecture [1] relates two important parameters of a graph, the chromatic number $\chi(G)$ and $h(G)$, the size of the largest clique minor of the graph. Specifically, it asserts that $\chi(G) \leq h(G)$ (or, equivalently, that any l-chromatic graph has K_l as a minor).

The conjecture has been proved in several special cases (e.g. [2]), and for random graphs [3]. Thus it is of interest to compare and prove various inequalities involving one of the two parameters.

Two situations are possible, described as follows:

Definition 1 *Let*

$$\chi(G) \leq w(G) \tag{1}$$

be an inequality, valid for graph parameter $w(G)$ and all graphs G from a certain class $\mathcal{C}$. We say that $h(G)$ interpolates in equation (1) *iff inequality*

$h(G) \leq w(G)$ is valid for all graphs $G \in \mathcal{C}$.

Of course, $h(G)$ might fail to interpolate in equation (1) for certain parameters $w(G)$.

Example 1 *Coffman, Hakimi and Schmeichel [4] have proved (as a corollary) the following upper bound on the chromatic number of a connected graph with m edges:*

$$\chi(G) \leq \frac{1}{2} + \frac{1}{4} \cdot \sqrt{\frac{1}{4} + 2m} \tag{2}$$

The inequality that shows that Hadwiger's number is interpolating in inequality (2) is stated explicitly and proved in [5].

Proving that a parameter is interpolating strengthens, of course, the corresponding inequalities for the chromatic number, provider Hadwiger's conjecture is true. Thus, in an intuitive sense, interpolating inequalities are weaker than non-interpolating ones, at least for special graphs, though it is, of course, possible that interpolating inequalities are tight in some cases

The purpose of this note is to study the question of interpolation of Hadwiger's number for several upper bounds on the chromatic number due to Ershov and Kozhukin [6], Coffman, Hakimi and Schmeichel [4], Brooks [7], Welsh and Powell [8], and Stacho [9,10].

2 Preliminaries

An useful reference for inequalities concerning the chromatic number is [11]. For Hadwiger's conjecture we refer the reader to [12].

We will use the standard notations K_r and C_r to denote the complete, respectively cycle graph with r vertices. A graph G is *1-reducible* to graph H if the iterative removal of vertices of degree one from G yields graph H.

Definition 2 *Let G be a graph and $e \in E(G)$ be an edge. The* contracted graph G_e *is obtained from G by identifying the endpoints of e and joining the newly created vertex with all vertices that were adjacent to either endpoint. All other vertices and edges are the same as in G.*

Definition 3 Hadwiger's number $h(G)$ *of connected graph G is the largest n such that one can obtain the complete graph K_n from G by a series of contractions.*

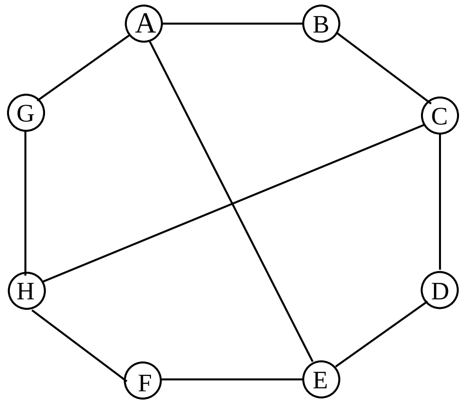

Fig. 1. The counterexample.

Definition 4 *For a graph G let $\Delta(G)$ be the largest degree of a vertex in G. Also define*

$$\Delta_2(G) = max_{u \in V(G)} max_{v \in N(u), d(v) \leq d(u)} d(v). \tag{3}$$

Definition 5 *For a graph G let V_i be the set of vertices of degree i. Also define*

$$s = s(G) = \max_{i \geq \frac{\Delta(G)+2}{2}} |V_i|. \tag{4}$$

3 Results

The following easy result shows that the upper bound on the chromatic number due to Ershov and Kuzhukin [6] is interpolating:

Theorem 1 *Let G be a connected graph with n vertices and m edges. Then*

$$h(G) \leq \lfloor \frac{3 + \sqrt{9 + 8(m-n)}}{2} \rfloor. \tag{5}$$

In [4] the following improved bound on the chromatic was proved: if G is a connected graph that is not 1-reducible to a clique or an odd cycle then $\chi(G) \leq \lfloor \frac{3+\sqrt{1+8(m-n)}}{2} \rfloor$. For Hadwiger's number the improved upper bound does not quite hold. A counterexample is presented in Figure 1 (with $n = 8$, $m = 10$, $h(G) = 4$). It consists of four trees (edges AB, CD, EF, and GH) with connections between all trees such that no vertex is left with degree one.

Nevertheless, we can recover the upper bound, making it interpolating by restricting the class of graphs it applies to:

Definition 6 *An* acyclic contraction *of graph G is a contraction that shrinks no cycle of G to a single edge.*

For example, no edge contraction in a complete graph K_n, $n \geq 3$ is an acyclic contraction, since it shrinks a triangle to a single edge. On the other hand any contraction in a tree is acyclic. In particular contracting edges AB, CD, EF, and GH in the graph in Figure 1 is an acyclic contraction to complete graph K_4.

Observation 1 *Any contraction specified by a 1-reducibility is acyclic.*

Observation 2 *If a graph G has an acyclic contraction to an odd cycle C_{2t+1} then (by additionally contracting parts of this cycle) one can obtain an acyclic contraction to $C_3 \equiv K_3$. Thus we don't really need odd cycles as a final target for acyclic contractions, avoiding acyclic contraction to a complete graph covers this case too.*

With these preliminaries we can state:

Theorem 2 *Let G be a connected graph with n vertices and m edges. If G does not have an acyclic contraction to a complete graph K_r then*

$$h(G) \leq \lfloor \frac{3 + \sqrt{1 + 8(m-n)}}{2} \rfloor .$$

On the other hand, the following result shows that many upper bounds are non-interpolating:

Theorem 3 *Hadwiger's number is* not *interpolating in any of the following inequalities for the chromatic number:*

(1) $\chi(G) \leq \Delta(G) + 1$ *(Brooks [7])*
(2) If the degrees of vertices in G are (in decreasing order) $d_1 \geq d_2 \geq \ldots \geq d_n$ *then* $\chi(G) \leq \max_{1 \leq j \leq n} \min\{j, d_j + 1\}$ *(Welsh-Powell [8])*
(3) $\chi(G) \leq \Delta_2(G) + 1$ *(Stacho [9])*
(4) $\chi(G) \leq \lceil \frac{s}{s+1}(\Delta(G) + 2) \rceil$ *(Stacho [10]),*

even when restricted to the class of 3-colorable graphs.

4 Proofs

4.1 Proof of Theorem 1

We first acknowledge the fact that we first made this simple observation in 1994, in a recreational mathematics journal for high-school students published in Romania. Since this journal is not accessible to the research community (in particular it is not indexed by any scientific database we know of, including AMS MathSciNet), and since the result contributes to the logic of the paper, we decided to present the proof here. Micu's proof [5] of the result stated in Example 1 uses a similar idea, but in the context of the weaker inequality (2).

Consider a partition of $V(G)$ into classes $V_1, V_2, \dots V_{h(G)}$ that correspond to vertices contracted to a specific vertex of $K_{h(G)}$. It is easy to see that the following are true:

(1) Each class V_i induces a connected subgraph of G.
(2) For every $i \neq j$ there exists at least one edge connecting a vertex in V_i to a vertex in V_j. This happens because the contracted graph $K_{h(G)}$ is a complete graph.

We have

$$m \geq \left(\sum_{i=1}^{h(G)} |E(V_i)|\right) + \binom{h(G)}{2} \geq \left(\sum_{i=1}^{h(G)} |V_i| - 1\right) + \binom{h(G)}{2} = \\ = [n - h(G)] + \binom{h(G)}{2}.$$

i.e.

$$m \geq n + \binom{h(G)}{2} - h(G). \tag{6}$$

Thus $h^2(G) - 3h(G) - 2(m-n) \leq 0$, and the result immediately follows. □

4.2 Proof of Theorem 2

To obtain the result we have to prove the inequality

$$m \geq n + 1 + \binom{h(G)}{2} - h(G) \tag{7}$$

Indeed, in the decomposition of graphs corresponding to obtaining $K_{h(G)}$ at least one of the following situations holds:

(1) At least one component corresponding to a vertex is *not* a tree (including single points among trees), or
(2) There exist two components connected by at least two edges.

Indeed, suppose neither 1 nor 2 applies. Then by contracting each tree component we would get an acyclic contraction to a $K_{h(G)}$ (including, possibly, the odd cycle C_3).

Thus m counts at least one extra edge not counted on the right hand side of equation 6, so equation 7 holds.

Now an argument similar to that in the proof of Theorem 1 completes the proof. □

4.3 Proof of Theorem 3

For $r \geq 3$ we construct a graph D_r with r^2 nodes, described as follows:

(1) Nodes are labelled 0 to $r^2 - 1$.
(2) For all $1 \leq i \leq r-1$, $0 \leq i \leq r-1$ we connect node $r \cdot i + j$ to node $r \cdot (i-1) + j$.
(3) For all $0 \leq i \neq j \leq r-1$ we connect nodes $r \cdot i + j$ and $r \cdot j + i$.

Graph D_3 is presented as an example in Figure 2.

For all $0 \leq j \leq r-1$ we can contract all nodes $r \cdot i + j$, $1 \leq i \leq r-1$ into a single node $\hat{j}$, by successively contracting edges (of type (ii)) $(r \cdot (i-1) + j, r \cdot i + j)$.

The contracted graph has r nodes, and any two of its nodes $\hat{i} \neq \hat{j}$ are connected, because of the edge between nodes $r \cdot i + j$ and $r \cdot j + i$ in graph D_r. Thus the contracted graph is the complete graph K_r, thus $h(D_r) \geq r$.

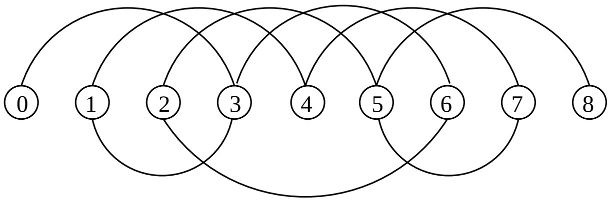

Fig. 2. The construction of graph D_r (here $r = 3$).

One can verify that, because of the way graph D_r is constructed $d(v) \leq 3$ for every $v \in V(G)$. Thus $\Delta(D_r) + 1 = 4$, $\Delta_2(D_r) + 1 = 4$ as well. As for the last bound, it is certainly no larger than $\Delta(D_r) + 2 = 5$.

Graph D_r is 3-colorable, for any $r \geq 3$, because it is *2-degenerate*: in the natural ordering of the vertices each vertex is adjacent to at most two earlier nodes. Thus the GREEDY algorithm 3-colors D_r. $\square$

5 Interpolating versus non-interpolating bounds revisited

Let us consider the question of obtaining interpolating upper bounds on Hadwiger's number in a graph where we know the degree sequence $d_1 \geq d_2 \geq \ldots \geq d_n$. Our results suggest an interesting difference between the interpolating and non-interpolating upper bounds: in the upper bounds in Theorem 1 and 2 the upper bounds depends on the degree sequence through quantity $m - n = \sum_i (d_i - 1)$. On the other hand the non-interpolating upper bounds are "local": they essentially depend only on one of the degrees (chosen according to the specific condition of each inequality).

Of course, one can transform the sequence $\sum_i (d_i - 1)$ to a "local" upper bound $n \cdot (\Delta(G) - 1)$. The question remains though, whether one can formalize the notion of "local" upper bounds in a way that excludes these trivial examples, and such that all interpolating upper bounds are *not* local (depend on $\Omega(n)$ terms in the sequence d_i). In a certain sense this should be true, given that interpolating inequalities in Theorem 1 and 2 are tight for all pairs (m, n), $n < m < \binom{n}{2}$ (see [4]). But we would like to see it better formalized.

6 Acknowledgments

This work has been supported by a PN II/"IDEI" grant from the Romanian CNCSIS.

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