Quasi-nonexpansivity and the convex feasibility problem

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Abstract. The weak and strong convergence of a sequence generated by the Mann-type iteration are investigated in a real Hilbert space framework. Some applications to the projection method for the convex feasibility problem are given. The key for the strong convergence in a Hilbert space is a property concerning the intersection of a family of convex closed sets (Lemma 1).

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1 Introduction

Let $\mathcal{H}$ be a real Hilbert space, let C be a convex closed subset of $\mathcal{H}$ and let $T: C \to C$ be a nonlinear mapping. Suppose that the set of fixed points of T in C, $Fix(T)$, is nonempty. There are various classes of mappings for which an element from $Fix(T)$ can be iteratively approximated by the *Mann-type* iteration process (sometimes the term *Krasnoselski/Mann* iteration is used as well). One of the most important such class is the class of *quasi-nonexpansive* mappings, introduced by Tricomi [26] for real valued functions and widely studied in the literature for more general cases [12, 23, 24].

Definition 1. *The mapping T is said to be quasi-nonexpasive (QNE) on C if*

$$\|T(x) - x^*\|^2 \leq \|x - x^*\|^2, \ \forall x \in C, \ x^* \in Fix(T).$$

The more general class of *demicontractive* mappings [16], [21], which properly includes the class of quasi-nonexpansive mappings, is often more desirable.

Definition 2. *The mapping T is said to be demicontractive (DC) if there exists a constant $k \in [0, 1)$ such that*

$$\|T(x) - x^*\|^2 \leq \|x - x^*\|^2 + k\|x - T(x)\|^2, \ \forall x \in C, \ x^* \in Fix(T). \qquad (1)$$

In the paper [20] was considered a class of mappings which satisfies the following condition: There exists a strictly positive number λ such that

$$\langle x - T(x), x - x^* \rangle \geq \lambda \|x - T(x)\|^2, \ \forall x \in C, \ x^* \in Fix(T). \tag{2}$$

C. Moore [21] observed that the class of maps satisfying this condition coincide with the class of demicontractive mappings. Indeed, it can be seen that (1) is equivalent with (2), where $\lambda = \frac{1-k}{2}$.

In [15] the more restrictive class of *firmly nonexpansive* (FNE) mappings satisfying the condition

$$\|T(x) - T(y)\|^2 \leq \|x - y\|^2 - \|x - y - (T(x) - T(y))\|^2, \ \forall x, y \in C, \tag{3}$$

was considered. For such mappings the simple iteration $x_{k+1} = T(x_k)$ it is shown that converges weakly to a fixed point of T for any initial iteration x_0, if such a fixed point exists. If the condition (3) is weaken requiring that $y \in Fix(T)$, then we obtain an other class of mappings:

Definition 3. *The mapping T is said to be firmly quasi-nonexpansive ($FQNE$) if*

$$\|T(x) - x^*\|^2 \leq \|x - x^*\|^2 - \|x - T(x)\|^2, \ \forall x \in C, \ x^* \in Fix(T). \tag{4}$$

Recently Combettes and Pennanen [11] have introduced a class of mappings satisfying the condition

$$\langle x^* - T(x), x - T(x) \rangle \leq 0, \ \forall x \in C, \ x^* \in FixT. \tag{5}$$

For such class it is shown [11] that the sequence generated by an iterative scheme of Mann-type converges weakly to a common fixed point of a family of mappings in this class. It can be seen that the class of mapping satisfying (5) coincides with the class of firmly quasi-nonexpansive mappings.

It is obvious that the considered classes of mappings satisfy the strictly inclusion relations:

$$FNE \subset FQNE \subset QEN \subset DC.$$

The *demiclosedness at zero* is an other notion frequently used in the studying of the Mann-type iteration.

Definition 4. *A maping T is said to be demiclosed at zero, if for any sequence $\{x_k\}$ which converges weakly to z, and if the sequence $\{T(x_k)\}$ converges strongly to zero, then $T(z) = 0$.*

In what follows we will consider the following Mann-type iterative scheme:

$$x_{k+1} = (1 - t_k)x_k + t_k T(x_k), \ x_0 \in C, \tag{6}$$

where $t_k \in R, \ k = 0, 1, ...$; usually, $0 < t_k < 1$ and then the next iteration x_{k+1} is a weightened mean value between the current iteration and the value of T in the current iteration. In particular, if we set $t_k = \frac{1}{2}$ then (6) becomes $x_{k+1} = (x_k + T(x_k))/2$, which is the well known Krasnoselski method. Note that the genuine Mann iteration [19] have the form $x_{k+1} = T(\bar{x}_k)$, where $\bar{x}_k$ denotes a convex combination of the points $\{x_j\}_{0\leq j\leq k}$. In [11] is pointed out the reason for which the iteration (6) is commonly referred as Mann iteration. Note also that the condition $0 < t_k < 1$ is not always satisfied, the typical example being the projection method in convex feasibility problems, for which t_k must satisfy the condition $0 < t_k < 2$.

In [20] the weak convergence of the sequence $\{x_k\}$ generated by (6) is shown, provided that $I - T$ is demiclosed at zero and the condition (2) is satisfied. For the strong convergence of the same sequence, additional condition concerning the structure of T and the starting iteration point is needed. These results were generalized to a Banach space and to an uniformly smooth Banach space in [9], [10], [27]. The almost identic conditions were used in [21] for proving the weak convergence of a Mann and Ishikawa iteration processes with errors to a fixed point of T, processes considered earlier in [18] and [28] for nonlinear strongly accretive operators.

The projection algorithms for solving the convex feasibility problem is a particular Mann-type iteration, having the form (6). The geometric idea of the method is to draw a normal line from the current iteration onto certain set form the intersection family and to take the next iteration on this line. A weight factor gives the exact position of the next iteration. The projection algorithm was used (it seems for the first time) in [1, 22] for solving a system of linear inequalities (the authors referred their method as "relaxation algorithm"). Generalizations for convex sets in real n-dimensional spaces were given in [13, 17]. Bergman [8] considered the classical projection method for the case of m intersecting closed convex sets M_i in a real Hilbert space. He showed that, given an arbitrary starting point x_0, the sequence generated by the projection algorithm converges weakly to a point in $M = \cap_{i=1}^{m} M_i$. In [14] certain regularity conditions on the sets were described that guaranteed strong convergence of the iterations. In some recent papers, other conditions for strong convergence have been given, for example in [6, 5, 11, 7]. A complete and exhaustive study on algorithms for solving convex feasibility problem, including comments about their applications and an excellent bibliography, was given by H.H.Bausche and J.M.Borwein [6].

In the section 2 a general theorem about the strong convergence of the simple iteration $x_{k+1} = T(x_k)$ for quasi-nonexpansive mappings is given. Section 3 deals with the weak convergence of the Mann iteration in Hilbert spaces and with the strong convergence of the same iteration in finite dimensional space. Finally, the projection method for convex feasibility problem is considered in section 4.

2 The strong convergence of the simple iteration

Let $d(x, E)$ denotes the distance between a point $x \in \mathcal{H}$ and a set $E \subset \mathcal{H}$, that is $d(x, E) = \inf_{y\in E} \|x - y\|$.

We shall use the following general theorem concerning the convergence of the simple iterates for quasi-nonexpansive mappings.

Theorem 1. *Suppose that $T : D \subset \mathcal{H} \to \mathcal{H}$ is a quasi-nonexpansive mapping and that $Fix(T)$ is nonempty and closed. Let $x_0 \in D$ such that $x_k = T_{x_0}^k \in D$,*

$k = 1, 2, \cdots$. Then the sequence $\{x_k\}$ converges (strongly) to a fixed point of T if and only if there exists a subsequence $\{x_{k_j}\}$ of $\{x_k\}$ such that $d(x_{k_j}, Fix(T)) \to 0$ as $j \to \infty$.

Here, as usual, T^k denotes the k iterate of T.

Remark 1. Theorem 1 is a slight generalization of the first result of [24] and its proof is similar. Essentially, Theorem 1 replaced the condition of continuity of T, from the original result, by the condition of closedness of $Fix(T)$. It is easy to see that the latter condition is weaker, and, as it will result, is essential for our development.

3 The weak and strong convergence of the Mann iteration

The sequence of scalars $\{t_k\}$ from (6) usually belong to the interval $(0, 1)$, and therefore, taking into account that C is convex and that $T : C \to C$, it follows that the whole sequence $\{x_k\}$ belongs to C. However, these scalars depend of the constant λ from (2) [20] which have no restrictions and so, the belonging of the sequence to C must be enforced as a condition.

Theorem 2. *Let $T : C \to C$ be a nonlinear mapping, where C is a closed convex subset of $\mathcal{H}$. Suppose that T is demicontractive on C, that $I - T$ is demiclosed at zero and that $a \le t_k \le b$, where a and b are some constants satisfying $0 < a, b < 1 - k$. Then the sequence $\{x_k\}$ generated by the Mann iteration converges weakly to an element of $Fix(T)$.*

Proof. Using the condition of demicontractivity (1) it obtains

$$\|x_{k+1} - x^*\|^2 \le \|x_k - x^*\|^2 - t_k(1 - k - t_k)\|x_k - T(x_k)\|^2.$$

Since $1-k-t_k > 0$, it follows that $\|x_{k+1}-x^*\|^2 \le \|x_k-x^*\|^2$ and so $\|x_k-x^*\| \to \rho_{x^*}$, as $k \to \infty$ for all $x^* \in Fix(T)$. Now, because $a \le t_k \le b$, it follows

$$\|x_k - T(x_k)\|^2 \le (a(1 - k - b))^{-1}(\|x_k - x^*\|^2 - \|x_{k+1} - x^*\|^2) \to 0 \ (k \to \infty).$$

The sequence $\{x_k\}$ being bounded, there exists a subsequence $\{x_{k_j}\}$ of $\{x_k\}$ which converge weakly to an x^*; since $\{x_{k_j}\} \subset C$ and C is closed and convex (hence weakly closed), it follows that $x^* \in C$. Moreover, x^* is a fixed point of T, for $x_{k_j} - T(x_{k_j}) \to 0$ and $I - T$ is demiclosed at zero (hence $x^* - T(x^*) = 0$).

Suppose there are two subsequence of $\{x_k\}$, say $\{u_k\}$ and $\{v_k\}$, which converge weakly to u and v, respectively. As above, we have that u and v are in $Fix(T)$ and that,

$$\|x_k - u\| \to \rho_u, \quad \|x_k - v\| \to \rho_v. \tag{7}$$

Now, consider the sequence

$$e_k = \|u_k - u\|^2 - \|v_k - u\|^2 - \|u_k - v\|^2 + \|v_k - v\|^2.$$

Since the relations (7) hold for any subsequence of $\{x_k\}$ (in particular for $\{u_k\}$ and $\{v_k\}$), it follows that $e_k \to 0$ as $k \to \infty$. On the other hand, by a simple computation, it obtains

$$e_k = -2\langle u_k - v_k, u - v\rangle.$$

This and the weak convergence of $\{u_k\}$ and $\{v_k\}$ to u and v , respectively, imply that $e_k \to -2\|u - v\|^2$ and, hence, $u = v$. Therefore, all weakly convergent subsequence of $\{x_k\}$ have the same weak limit, say x^*. It follows that $\{x_k\}$ converge weakly to this fixed point. □

Remark 2. The proof follows almost verbatim the proof of Theorem 1 from [20]; we have reproduced it here because our idea of proof was used in certain recent papers [2–4]

From the applications point of view, it is interesting to obtain additional conditions such that the sequence $\{x_k\}$ converges strongly to an element of $Fix(T)$. In [25] the following condition is considered: There exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for $r > 0$, such that $\|x - T(x)\| \geq f(d(x, Fix(T)))$ for $\forall x \in C$. The following theorem gives also such a condition.

Theorem 3. *Let T be as in Theorem 2. If, in addition, there is $h \in C,\ h \neq 0$, such that $\langle x - T(x), h\rangle \leq 0$ for all $x \in C$, then the sequence $\{x_k\}$ generated by (6) with t_k also as in Theorem 2, and for suitable x_0 in C, converges strongly to an element of $Fix(T)$.*

The proof is in [20]. Note that suitable initial points x_0 are those that satisfies $\langle x_0 - x^*, h\rangle > 0$.

4 The convex feasibility problem

Let $M_i \subset H,\ i = 1, ..., m$ be a family of convex closed subsets of $\mathcal{H}$ with nonempty intersection, $\bigcap M_i \neq \emptyset$. The convex feasibility problem is:

$$Find\ a\ point\ of\ \bigcap M_i.$$

Let x be a point in $\mathcal{H}$ and let $P(x, i)$ be the projection of x onto M_i (if $x \in M_i$, then $P(x, i) = x$). Let i_x be the least index such that

$$\|x - P(x, i_x)\| = \max_i \|x - P(x, i)\|.$$

Define the mapping $T : \mathcal{H} \to \mathcal{H}$ by $T(x) = P(x, i_x)$. It is clear that $x \in \bigcap M_i$ if and only if $T(x) = x$, hence if and only if x is a fixed point of T, that is $\bigcap M_i = Fix(T)$. For any $x \in \mathcal{H}$ and $x^* \in Fix(T)$, it has that $\langle x - P(x, i_x), P(x, i_x) - x^*\rangle \geq 0$ and it is routine to see that T is firmly quasi-nonexpansive.

In the case of a finite dimensional Hilbert space $\mathcal{H} = \mathcal{R}^n$ (in particular an Euclidean space), $I - T$ is demiclosed at zero. Indeed, in a finite dimensional space the weak and strong convergence coincide and let $\{x_k\}$ be a sequence such that $x_k \to x^*$ and $x_k - T(x_k) \to 0$ as $k \to \infty$. For each i, $(1 \le i \le m)$ it has

$$\|x_k - P(x_k, i)\| \le \|x_k - P(x_k, i_{x_k})\| = \|x_k - T(x_k)\| \to 0,\ k \to \infty.$$

Since $P(x, i)$ is a continuous function for each i it follows

$$\lim_{k\to\infty} \|x_k - P(x_k, i)\| = \|x^* - P(x^*, i)\| = 0,$$

for each i. Therefore $x^* - T(x^*) = x^* - P(x^*, i_{x^*}) = 0$ and so $I - T$ is demiclosed at zero. The theorem 2 may be applied and it results

Theorem 4. *The sequence $\{x_k\}$ generated by*

$$x_{k+1} = (1 - t_k)x_k + t_k P(x_k, i_{x_k}),\ x_0 \in R^n,$$

with $0 < a \le t_k \le b < 2$, converges to an element of $\bigcap M_i$.

Remark 3. Theorem 4 is due to I.I. Eremin [13]; when M_i is defined by a system of linear inequalities, that is each set M_i is a half space, then it obtains a more special case developed in [1, 22].

Remark 4. It is easy to see that $T(x) = P(x, i_x)$ is discontinuous in those points $x \in \mathcal{R}^n$ where $max\|x - P(x, i)\|$ is touched for more than one value of index i.

In the case of a general Hilbert space H it needs an additional condition for the strong convergence of the Mann iteration. The following lemma point out such a condition.

Lemma 1. *Let $M_i \subset \mathcal{H}$ $(i = 1, \cdots, m)$ be a family of convex sets such that $Int \bigcap M_i$ is nonempty and bounded and let $\{x_k\}$ be a sequence of $\mathcal{H}$ such that $d(x_k, M_i) \to 0$ as $k \to \infty$ for each i. Then $d(x_k, \bigcap M_i) \to 0$, as $k \to \infty$.*

Proof. We assume that $o \in Int \bigcap M_i$. Then there exists a closed ball $D(o, r) = \{x \in \mathcal{H} : \|x\| \le r\} \subset \bigcap M_i$. Let ϵ be a given real number, $0 < \epsilon < 1$, and let C be a constant such that $\|x\| \le C - 1$ for all $x \in \bigcap M_i$, which is possible, because $\bigcap M_i$ is bounded.

Since $d(x_k, M_i) \to 0$ as $k \to \infty$, for each index i, there exists a sequence $\{y_k^{(i)}\}_{k \in N} \subset M_i$ such that $\|y_k^{(i)} - x_k\| \to 0$ as $k \to \infty$. Let

$$z_k = (1 - \frac{C}{\epsilon})(y_k^{(i)} - x_k), k = 0, 1, \cdots. \tag{8}$$

There exists a number $k_i(\epsilon)$ such that if $k \ge k_i(\epsilon)$ then $\|y_k^{(i)}\| \le \frac{r}{|1 - \frac{C}{\epsilon}|}$ and so $\|z_k\| \le r$, that is $z_k \in \bigcap M_i$.

On the other hand, from (8) we obtain

$$(1 - \frac{\epsilon}{C})x_k = \frac{\epsilon}{C} z_k + (1 - \frac{\epsilon}{C}) y_k^{(i)},$$

and for $k \geq k_i(\epsilon)$ we have $(1-\frac{\epsilon}{C})x_k \in M_i$, because $y_k^{(i)}$, $z_k \in M_i$ and M_i are convex.

Now, let $k_0(\epsilon) = \max_i k_i(\epsilon)$. Then, for $k \geq k_0(\epsilon)$ it follows that $(1-\frac{\epsilon}{C})x_k \in \bigcap M_i$ and

$$d(x_k, \bigcap M_i) \leq \|x_k - (1-\frac{\epsilon}{C})x_k\| = \frac{\epsilon}{C-\epsilon}\|(1-\frac{\epsilon}{C})x_k\| < \epsilon,$$

which end the proof. □.

Apparently, the condition that $Int \bigcap M_i$ is nonempty and bounded is very strong. The following example shows that this condition cannot be replaced by the weaker condition $\bigcap M_i \neq \emptyset$, which seems to be more natural.

Example. Suppose that $\mathcal{H}$ is the real three-dimensional space, that $m = 2$, that the set M_1 is a cone (A) and the set M_2 is a tangent plane $(ABCD)$. The situation is depicted in Figure 1

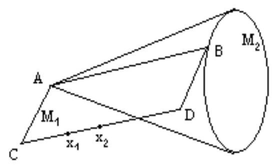

Fig. 1. Example

The plane $(ABCD)$ is tangent to the cone along the generatrix (AB) and hence $M_1 \bigcap M_2 = (AB)$. Now, let us consider a sequence $\{x_k\}$ in the plane $(ABCD)$ such that $d(x_k, (AB)) = \delta = const.$ and $\|x_k\| \to \infty$ as $k \to \infty$. It is clear that $d(x_k, M_2) \to 0$ as $k \to \infty$ and $d(x_k, M_1) = 0$ for all k; but $d(x_k, M_1 \bigcap M_2) = \delta > 0$. Therefore, the conclusion of Lemma 1 is not true.

In the following we consider a particular case of (6) for which $t_k = \lambda,\ k = 0, 1, ...$ and $\lambda \in (0, 2)$. The Mann iteration is defined by the iteration function $T_\lambda = I - \lambda(I - T)$, where T is defined above by $T(x) = P(x, i_x)$. Obviously, $Fix(T) = Fix(T_\lambda)$.

Theorem 5. *Let M_i $(i = 1, \cdots, m)$ be a family of closed convex sets of $\mathcal{H}$ such that $Int \bigcap M_i$ is nonempty and bounded. Then the sequence $\{x_k\}$ given by $x_{k+1} = T_\lambda^k(x_0)$ converges strongly to a point of $\bigcap M_i$ for all $x_0 \in \mathcal{H}$.*

Proof. Since $Fix(T_\lambda) = \bigcap M_i$ is a closed set, it suffices to show that T_λ is quasi-nonexpansive on $\mathcal{H}$ and that $d(x_k, \bigcap M_i) \to 0$ as $k \to 0$. Then Theorem 5 follows from Lemma 1 and Theorem 1.

Let $x \in \mathcal{H}$ and $y \in \bigcap M_i$. Since $P(x, i_x)$ is the projection of x onto M_{i_x} and $x^* \in M_{i_x}$, we have

$$\langle T(x) - x^*, x - T(x)\rangle = \langle P(x, i_x) - x^*, x - P(x, i_x)\rangle \geq 0,$$

and

$$\|T_\lambda(x) - x^*\|^2 \le \|x - x^*\|^2 - \lambda(2-\lambda)\|x - T(x)\|^2. \tag{9}$$

Therefore, we have

$$\|T_\lambda(x) - x^*\| \le \|x - x^*\|, \forall x \in \mathcal{H}, x^* \in \bigcap M_i, \tag{10}$$

and T_λ is quasi-nonexpansive on $\mathcal{H}$.

Now, since $x_{k+1} = T_\lambda(x_k)$, from (10) it follows that the sequence $\{\|x_k - x^*\|\}$ is monotone decreasing and bounded, therefore $\|x_k - x^*\| \to \delta_y$ as $k \to \infty$, for each $y \in \bigcap M_i$. From (9) we obtain

$$\|x_k - T(x_k)\|^2 \le \frac{1}{\lambda(2-\lambda)}(\|x_k - y\|^2 - \|x_{k+1} - y\|^2)$$

and hence $\|x_k - T(x_k)\| \to 0$ as $k \to \infty$. But $\|x - P(x,i)\| \le \|x - T(x)\|$ for each i. Therefore $d(x_k, M_i) = \|x_k - P(x_k, i)\| \to 0$ as $k \to \infty$ and Theorem 5 is proved $\square$.

Remark 5. It is easy to see that the mapping $T : \mathcal{H} \to \mathcal{H}$ defined above ($T(x) = P(x, i_x)$) is not throughout continuous. Indeed, let $m = 2$ and let x be a point of $\mathcal{H}$ such that $d(x, M_1) = d(x, M_2)$. Now, let $\{x_k\}$ be a sequence such that $x_k \to x$ as $k \to \infty$ and $d(x_k, M_1) < d(x_k, M_2)$ for all k. Then $\lim Tx_k = \lim P(x_k, M_2) = P(x, M_2)$; but $T(x) = P(x, M_1)$, this means that T is not continuous at x.

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